

AUTOMORPHISMS OF SINGULAR THREE-DIMENSIONAL CUBIC HYPERSURFACES

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ABSTRACT. In this paper we classify three-dimensional G -del Pezzo varieties of degree 3, which are not G -rational and have no birational structure of G -Mori fiber space with the base of positive dimension.

1. INTRODUCTION

In this paper we work over an algebraically closed field k of characteristic 0. Recall that a G -variety is a pair (X, ρ) , where X is an algebraic variety and $\rho : G \rightarrow \text{Aut}(X)$ is an injective homomorphism of groups. We say that G -variety X has $G\mathbb{Q}$ -factorial singularities if every G -invariant Weil divisor of X is \mathbb{Q} -Cartier.

Let X be a G -variety with at most $G\mathbb{Q}$ -factorial terminal singularities and $\pi : X \rightarrow Y$ be a G -equivariant morphism. We call π a G -Mori fibration if $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, $\dim X > \dim Y$, the relative invariant Picard number $\rho^G(X/Y)$ is equal to 1 (in this case we say that G is *minimal*) and the anticanonical class $-K_X$ is π -ample. If Y is a point then X is a $G\mathbb{Q}$ -Fano variety. If in addition the anticanonical class is a Cartier divisor then X is a G -Fano variety.

Let X be arbitrary normal projective G -variety of dimension 3. Resolving the singularities of X and applying the G -equivariant minimal model program we reduce X either to a G -variety with nef anticanonical class, or to a G -Mori fibration (see e.g. [18, §3]). So such fibrations (and $G\mathbb{Q}$ -Fano varieties in particular) form a very important class in the birational classification. In this paper we consider a certain class of G -Fano threefolds.

Definition 1.1. A projective n -dimensional variety X is a *del Pezzo variety* if it has at most terminal Gorenstien singularities and the anticanonical class $-K_X$ is ample and divisible by $n - 1$ in the Picard group $\text{Pic}(X)$. If a G -Fano variety X is a del Pezzo variety, then we say that X is a G -del Pezzo variety.

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Del Pezzo varieties of arbitrary dimension were classified by T. Fujita ([14], [13], [15]). Three-dimensional G -del Pezzo varieties were partially classified by Yu. Prokhorov in [20]. The main invariant of a del Pezzo threefold X is the *degree* $d = (-\frac{1}{2}K_X)^3$, it is an integer in the interval from 1 to 8. In this paper we consider the case $d = 3$. In this case X is a cubic hypersurface in \mathbb{P}^4 . The case $d = 4$ was considered in our previous work [2]. If $d > 4$ then X is smooth (cf. [20]) while smooth del Pezzo threefolds and their automorphism groups are known well. For other types of G -Fano threefolds there are only some partial results: see for example [22], [25].

Classification of finite subgroups of the Cremona group $\text{Cr}_3(k)$ is one of the motivations of this paper. The Cremona group $\text{Cr}_n(k)$ is the group of birational automorphisms of the projective space \mathbb{P}_k^n . Finite subgroups of $\text{Cr}_2(k)$ were completely classified by I. Dolgachev and V. Iskovskikh in [11]. The core of their method is the following. Let G be a finite subgroup of $\text{Cr}_2(k)$. The action of G can be regularized in the following sense: there exists a smooth projective G -variety Z and an equivariant birational morphism $Z \rightarrow \mathbb{P}^2$. Then we apply the equivariant minimal model program to Z and obtain a G -Mori fibration which is either a G -conic bundle over \mathbb{P}^1 (which is a blowing up of a Hirzebruch surface in some points), or a G -minimal del Pezzo surface. Dolgachev and Iskovskikh classified all minimal subgroups in automorphism groups of del Pezzo surfaces and conic bundles and so they obtained the full list of finite subgroups of $\text{Cr}_2(k)$. But quite often two subgroups from such list are conjugate in $\text{Cr}_2(k)$, so it is natural to identify them. One can see that G -varieties Z_1 and Z_2 give us conjugate subgroups if and only if there exists a G -equivariant birational map $Z_1 \dashrightarrow Z_2$. So we need to classify all rational G -Mori fibrations and birational maps between them as well.

Following this program in the three-dimensional case one can reduce the question of classification of all finite subgroups in $\text{Cr}_3(k)$ to the question of classification of all rational $G\mathbb{Q}$ -Mori fibrations and birational equivariant maps between them. Such program was realized in some particular cases: simple non-abelian subgroups (see [24], see also [6], [7], [8], [5]) and p -elementary subgroups of $\text{Cr}_3(\mathbb{C})$ (see [23], [19]).

For applications to Cremona groups we are mostly interested in classification of *rational* del Pezzo varieties, so we assume that X is singular (every smooth cubic threefold is not rational due to the classical result of Clemens and Griffiths [9]). Singular cubic threefolds whose singularities are nodes were classified by H. Finkelberg and J. Werner in [12]. They have described the mutual arrangement of singular points and planes, divisorial class groups and small resolutions of such varieties.

Following their paper [12], we will use the notation J1–J15 for nodal cubic threefolds everywhere below.

In this paper we are interested in the following problem: classify rational G -del Pezzo threefolds of degree 3 that have no G -equivariant birational map to a “more simple” G -Fano threefold (for example \mathbb{P}^3 or a quadric in \mathbb{P}^4) or to a G -Mori fibration with the base of positive dimension. In this paper we give a partial answer for this question.

In this paper we use the following notation:

- \mathfrak{C}_n is the cyclic group of order n ;
- \mathfrak{D}_{2n} is the dihedral group of order $2n$;
- \mathfrak{S}_n is the symmetric group of degree n ;
- \mathfrak{A}_n is the alternating group of degree n ;
- G^n is the direct sum of n copies of the group G ;
- $(a_1, a_2, \dots, a_{n_1})(a_{n_1+1}, \dots, a_{n_2}) \dots (a_{n_{m-1}+1}, \dots, a_{n_m})$ is a cyclic decomposition of an element $\sigma \in \mathfrak{S}_N$;
- $\mathfrak{S}_5 \subset \mathfrak{S}_6$ is a *standard* subgroup iff it is the stabilizer of some element in $\{1, \dots, 6\}$ with the natural action of \mathfrak{S}_6 , and *non-standard* otherwise. We use the same notation for $\mathfrak{A}_5 \subset \mathfrak{S}_6$;
- δ_j^i is the Kronecker symbol: $\delta_j^i = 1$ if $i = j$ and $\delta_j^i = 0$ otherwise.

Definition 1.2. We call a group $G \subset \text{Aut}(X)$ *linearizable* (resp., *of fiber type*) if there exists a G -equivariant birational map $X \dashrightarrow \mathbb{P}^3$ (resp., $X \dashrightarrow X'$ where X' has a structure of a G -Mori fibration with the base of positive dimension).

The main result of this paper is the following theorem:

Theorem 1.3. *Let $X = X_3 \subset \mathbb{P}^4$ be a cubic hypersurface and G be a finite subgroup of $\text{Aut}(X)$ such that X is a rational G -Fano variety. Suppose that G is neither linearizable nor of fiber type. Then all singularities of X are ordinary double points (nodes) and X is described in the following table:*

type ₁	type ₂	$r(X)$	$s(X)$	$\text{Aut}(X)$	G
J15	31°	6	10	\mathfrak{S}_6 acts by permutations of coordinates x_0, \dots, x_5 (see J15 below)	\mathfrak{A}_5 , \mathfrak{S}_5 (standard subgroups), \mathfrak{A}_6 , $\text{Aut}(X)$

J14	30°	5	9	$\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$ acts by permutations of coordinates x_0, \dots, x_5 (see J15 below)	\mathfrak{S}_3^2 (subgroup of $\text{Aut}(X)$ acting transitively on the set of coordinates), $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_4$, $\text{Aut}(X)$
J9a	28°	2	6	\mathfrak{S}_5	$\text{Aut}(X)$
J9b	28°	2	6	$\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$	\mathfrak{S}_3^2 (a unique subgroup acting transitively on $\text{Sing}(X)$), $\text{Aut}(X)$
J5a	27°	1	5	\mathfrak{S}_5	$\mathfrak{C}_5 \rtimes \mathfrak{C}_4$, \mathfrak{A}_5 , $\text{Aut}(X)$
J5b	27°	1	5	\mathfrak{A}_5	$\text{Aut}(X)$

where $r(X) = \text{rank Cl}(X)$, $s(X) = |\text{Sing}(X)|$, type_1 is a type of the variety in terms of [12] and type_2 is a type of the variety in terms of [20]. Moreover, in some coordinate system X has the following equation in these cases:

$$\text{J15: } \left\{ \sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0 \right\} \subset \mathbb{P}^5, \text{ i.e. } X \text{ is the Segre cubic;}$$

$$\text{J14: } \{x_0x_1x_2 - x_3x_4x_5 = \sum_{i=0}^5 x_i = 0\} \subset \mathbb{P}^5;$$

$$\text{J9a: } \sum_{i=0}^4 x_i x_{1+i} x_{2+i} - \sum_{i=0}^4 x_i x_{1+i} x_{3+i} = 0 \text{ (here we consider indices modulo 5);}$$

$$\text{J9b: } x_0x_1x_2 - x_0x_1x_3 + x_0x_1x_4 + x_0x_2x_3 - 3x_0x_2x_4 + x_0x_3x_4 - x_1x_2x_3 + x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4 = 0;$$

$$\text{J5a: } \sum_{0 \leq i < j < k \leq 4} x_i x_j x_k = 0;$$

$$\text{J5b: } \sum_{i=0}^4 x_i x_{1+i} x_{2+i} - \omega \sum_{i=0}^4 x_i x_{1+i} x_{3+i} = 0 \text{ where } \omega \text{ is a primitive cubic root of unity (both roots give us isomorphic varieties; here we consider indices modulo 5).}$$

Conversely, for any group G in the table above (X, G) is a G -Fano variety.

Remark 1.4. We do not state that all G -varieties from the table actually are not linearizable or not of fiber type. It requires much more detailed analysis of G -birational rigidity of every case separately.

2. SINGULARITIES OF THREE-DIMENSIONAL CUBIC HYPERSURFACES

Assumption 2.1. Throughout this paper X is a singular cubic hypersurface in \mathbb{P}^4 with only terminal singularities and G is a finite subgroup of $\text{Aut}(X)$ such that G is minimal and is neither linearizable nor of fiber type (so $\text{Aut}(X)$ also has such properties).

Remark 2.2. Since X is $G\mathbb{Q}$ -factorial, the G -minimality of X is equivalent to $\text{rk Cl}(X)^G = 1$.

Since the Picard group $\text{Pic}(X)$ is generated by the class of a hyperplane section (see [16, Corollary 4.3.2]), the action of G on X is induced from a linear action of G on \mathbb{P}^4 .

Lemma 2.3. *Let Y be a three-dimensional G -variety and let Z be a G -variety of dimension 1 or 2. Let $Y \dashrightarrow Z$ be a G -equivariant dominant rational map whose general fiber is either a rational curve or rational surface. Then there exists the following commutative diagram*

$$(1) \quad \begin{array}{ccc} Y & \dashrightarrow & Y' \\ \downarrow & & \downarrow \\ Z & \longleftarrow & Z' \end{array}$$

where $Y' \rightarrow Z'$ is a G -Mori fiber space and both horizontal maps are birational and G -equivariant.

Proof. Note that one can equivariantly compactify a G -variety S by compactifying the quotient S/G and taking its normalization in the function field $k(S)$. Then we may assume that Y and Z are projective. Then we can apply equivariant resolution of singularities (see [1]). So we may assume that $Y \dashrightarrow Z$ is a G -equivariant morphism between smooth projective varieties. Then we apply the G -equivariant relative minimal model program to Y over Z . Since fibers of the map $Y \rightarrow Z$ are rationally connected, in the end we obtain a required G -Mori fibration $Y' \rightarrow Z'$. This gives us the required commutative diagram. \square

Remark 2.4. In the case where $Y \dashrightarrow Z$ is of relative dimension one the commutative diagram (1) can be taken so that Y' and Z' are smooth varieties (see [3] for details).

The following easy lemma is very important:

Lemma 2.5. (i) *The variety X has no G -fixed singular points and no G -invariant lines;*
(ii) *There are no G -invariant planes in \mathbb{P}^4 .*

Proof. If X contains a fixed singular point then the projection from this point is a G -equivariant birational map to \mathbb{P}^3 . Therefore G is linearizable in this case. If X contains a G -invariant line then the projection from this line gives us a G -equivariant rational curve fibration, and in the case of G -invariant plane we get a G -fibration by quadric or cubic surfaces in \mathbb{P}^3 (the first case occurs exactly when the plane lies on X). Then we apply Lemma 2.3. Therefore G is of fiber type in these cases. \square

Corollary 2.6. *The G -orbit of a singular point of X has length at least 4.*

Definition 2.7. We say that points $p_1, \dots, p_k \in \mathbb{P}^n$ are *in general position* if no d of them lie in a subspace $\mathbb{P}^{d-2} \subset \mathbb{P}^n$ for every $d \leq n+1$.

The following facts about singular points of cubic threefolds are well-known.

Lemma 2.8. *Let $Y \subset \mathbb{P}^4$ be an arbitrary cubic hypersurface with isolated singularities. Then*

- (i) *no three singular points lie on one line;*
- (ii) *if four singular points lie on a plane then this plane is contained in Y and there is no other singular points on it;*
- (iii) *if no four singular points lie on a plane then all singular points of Y are in general position.*

Proof. The first statement can be easily deduced from the fact that the singular set of X is an intersection of quadrics.

Assume that four singular points of Y lie on a plane P . Suppose that P is not contained in Y . Then $Y \cap P$ is a cubic curve with four singular points such that no three of them lie on one line. It is impossible, so P lies on Y . The second part of (ii) also follows from the fact that the singular set of X is an intersection of quadrics.

Let H be a hyperplane such that it contains at least five singular points of Y and no four of them lie on a plane. Consider the intersection $Z = Y \cap H$. It is a cubic surface with at least five singular points. Due to [4] such a surface must be reducible. If Z is a union of a quadric and a plane, then at least four singular points lie on this plane. If Z is a union of three different planes then all singular points lie on three lines (each line is the intersection of two planes). Thus in this case we again see, that one of three planes contains four singular points of Y . If Z contains a double or triple plane, then such plane is the singular set of Z , so it contains five singular points of Y . This contradiction proves (iii). \square

Proposition 2.9. *Assume that all singularities of X are nodes. Then there is one of the following possibilities:*

type of X	J4	J5	J9	J11	J14	J15
$s(X)$	4	5	6	6	9	10
$p(X)$	0	0	0	3	9	15
$r(X)$	1	1	2	3	5	6

where $s(X)$ is the number of nodes, $p(X)$ is the number of planes on X and $r(X)$ is the rank of $\text{Cl}(X)$.

Proof. Cases J1–J3 of [12] are impossible because X has at least four singular points by Corollary 2.6. If X is of type J6, J7 or J8, then X contains exactly one plane, which is $\text{Aut}(X)$ -invariant. If X is of type J10 or J12, then there is a distinguished $\text{Aut}(X)$ -invariant singular point of X (p_7 more precisely). If X is of type J13, then there is a distinguished quadruple of singular points p_5, p_6, p_7, p_8 which lie on one $\text{Aut}(X)$ -invariant plane. Hence in all these cases we can apply Lemma 2.5 to obtain a contradiction. Numbers $s(X)$, $p(X)$ and $r(X)$ can be found in [12]. \square

Now we consider the case where not all the singularities of X are nodes.

Proposition 2.10. *Assume that X contains a singularity which is not a node. Then the variety X has exactly four or five singularities of type cA_1 or cA_2 in general position and no other singularities.*

Proof. We have the following formula for the degree of the dual variety:

$$\deg X^\vee = 3 \cdot 2^3 - \sum_{p \in \text{Sing}(X)} m(p),$$

where $m(p) = \mu(p) + \mu'(p)$ is the sum of Milnor numbers of singularity (X, p) and its hyperplane section (see [28] for details). Obviously, we have $\deg X^\vee \geq 3$, hence

$$\sum_{p \in \text{Sing}(X)} m(p) \leq 21.$$

Note that three-dimensional terminal hypersurface singularities are exactly cDV points (see [17, Corollary 5.38]). If p is a cDV singularity which is not of type cA_1 , then $\mu(p) \geq 2$ and $\mu'(p) \geq 2$ (one can see it easily from the definition of Milnor number), and equalities hold exactly for cA_2 singularities. Thus singularities are of type cA_2 and we have four or five of them. The number of remaining singularities

cannot be greater than $\frac{21-4\cdot 4}{2} = \frac{5}{2}$, so there are no other singularities on X .

Assume that all singularities of X are cA_1 points. In some analytic neighbourhood they can be given by an equation $x^2+y^2+z^2+t^n = 0$, $n > 1$. Hence they are *rs-nondegenerate* (see [20, Definition 10.1, Proposition 10.3]). On the other hand, $r(X) = \text{rk Cl}(X) \leq 3$ by [20, Theorems 1.7, 7.1 and 8.1]. Thus we can apply the following formula (see [20, Proposition 10.6]):

$$(2) \quad \sum_{p \in \text{Sing}(X)} \lambda(X, p) \leq r(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\widehat{X}) \leq 7,$$

where X' is a general smooth cubic, $\rho(X)$ is the Picard number of X , variety \widehat{X} is a standard resolution of X (see [20, Definition 10.1]) and $\lambda(X, p)$ is the number of exceptional divisors of $\widehat{X} \rightarrow X$ over p . It follows from (2) and Corollary 2.6 that all singular points lie in one $\text{Aut}(X)$ -orbit and $\lambda(X, p) = 1$ for every singular point p . By the direct computations one can easily see that $\lambda(X, p) = 1$ iff $n \leq 3$. Hence $n = 3$. Assume that the number of singular points is greater than 5. Then $r(X) \geq 2$ by the formula (2) and X is not \mathbb{Q} -factorial. But the singularity $x^2 + y^2 + z^2 + t^3 = 0$ is \mathbb{Q} -factorial (see [26, Corollary 1.16]), a contradiction. Thus the number of cA_1 points which are not nodes cannot be greater than 5.

As an easy consequence from the Lemma 2.8, all singular points are in general position. \square

Remark 2.11. Later we will show (see §7 and §8) that two cases from the statement of the Proposition 2.10 cannot occur.

3. SEGRE CUBIC

In this section we consider the case where X is a cubic hypersurface in \mathbb{P}^4 of type J15 satisfying Assumption 2.1 and G is the corresponding minimal subgroup of $\text{Aut}(X)$. Such a variety is unique up to isomorphism. It is called *Segre cubic* and can be explicitly given by the following system of equations:

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0$$

in \mathbb{P}^5 . This variety has many interesting properties (see e.g. [10, §9.4.4]):

- This is a unique cubic threefold with ten isolated singularities.
- The automorphism group of X is isomorphic to \mathfrak{S}_6 and acts on X via permutation of coordinates.

- The variety X contains exactly 15 planes forming an $\text{Aut}(X)$ -orbit and one of them is given by equations

$$x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0.$$

- Singular points of X form an $\text{Aut}(X)$ -orbit, one of them has coordinates $(1 : 1 : 1 : -1 : -1 : -1)$.

Proposition 3.1. *The group G coincides with one of the following subgroups of $\text{Aut}(X) \simeq \mathfrak{S}_6$:*

$$\mathfrak{A}_5, \mathfrak{S}_5, \mathfrak{A}_6, \mathfrak{S}_6,$$

where \mathfrak{S}_5 and \mathfrak{A}_5 are standard subgroups.

Proof. One can easily prove this using the list of all subgroups of \mathfrak{S}_6 (for example, one can use GAP [29] to construct such a list) and following simple facts:

- The group G with the natural action on the set of coordinates cannot have an invariant pair of coordinates: otherwise there is a G -invariant plane in \mathbb{P}^4 .
- The group G is not a subgroup of $\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2 \subset \text{Aut}(X)$ since such group is a stabilizer of some singular point of X (see [21, p. 252]).
- Every subgroup $\mathfrak{S}_5 \subset \text{Aut}(X)$ acting transitively on the set of coordinates has an orbit of length 5 in the set of planes in X (see [21, Lemma 3.8]). The sum of all planes in such orbit cannot be proportional to the canonical class of X . Hence such subgroups \mathfrak{S}_5 are not minimal. As a consequence, G is not a subgroup of such group.
- Every subgroup $H \simeq \mathfrak{S}_4 \times \mathfrak{C}_2$ acting transitively on the set of coordinates is the stabilizer of some plane on X since all such subgroups are conjugate and the stabilizer of a plane has the same property. As a consequence, G is not a subgroup of such group.

Using these facts we exclude all the possibilities for G except four classes from the assertion. On the other hand all these four classes of subgroups act transitively on the set of planes, so they are minimal. \square

Remark 3.2. I.Cheltsov and C. Shramov [6] proved that the Segre cubic with the action of $G = \mathfrak{A}_6$ has no equivariant birational transformations to another G -Mori fibration (in other words, the Segre cubic is G -birationally rigid). So the G -birational rigidity of Segre cubic with the action of \mathfrak{A}_5 and \mathfrak{S}_5 is the only open problem related to the G -rigidity property of the Segre cubic.

4. CUBIC HYPERSURFACES WITH NINE SINGULAR POINTS

In this section we consider the case where X is a cubic hypersurface in \mathbb{P}^4 of type J9 satisfying Assumption 2.1 and G is the corresponding minimal subgroup of $\text{Aut}(X)$. In this case X is a hyperplane section of the cubic fourfold $Z \subset \mathbb{P}^5$, which can be explicitly given by the following equation: $x_1x_2x_3 = y_1y_2y_3$ (see [27, Prop. 2.2]). One can easily see that the group $\text{Aut}(Z)$ is isomorphic to $(k^*)^4 \rtimes (\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2)$ where the algebraic torus $(k^*)^4$ acts on coordinates diagonally and the subgroup $\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$ acts naturally by permutation of coordinates (see [27, Lemma 1.1]). The singular locus $\text{Sing}(Z)$ consists of nine lines

$$l_{ij} = \{x_k = y_l = 0, k \neq i, l \neq j\}$$

which intersect X in the singular locus of X . Also Z contains nine 3-spaces

$$M_{ij} = \{x_i = y_j = 0\}$$

and the intersections of these subspaces with X are planes on X . Note that there is a natural $\text{Aut}(X)$ -equivariant bijection $l_{ij} \leftrightarrow M_{ij}$ between lines and 3-spaces on Z .

Suppose that

$$a_1x_1 + a_2x_2 + a_3x_3 + b_1y_1 + b_2y_2 + b_3y_3 = 0$$

is an equation of the hyperplane which cuts out X . Notice that $a_i \neq 0$ and $b_i \neq 0$ for all i . Indeed, otherwise some lines from $\text{Sing}(Z)$ meet X at the same point, this is impossible. Using some diagonal coordinate change one can achieve $a_1 = a_2 = a_3 = A$, $b_1 = b_2 = b_3 = 1$ for some number A . One can easily see that if $A_1^3 = A_2^3$, then corresponding hyperplane sections are isomorphic. Note that if $A = -1$ then $(1 : 1 : 1 : 1 : 1 : 1)$ is a singular point of X and in fact X is the Segre cubic. Moreover, if $A^3 \neq -1, 0$ then X contains 9 nodes and no other singularities.

Lemma 4.1. *Let $Y = Z \cap H$ be a hyperplane section of Z such that Y is a cubic hypersurface in \mathbb{P}^4 with exactly 9 nodes as singularities. Then every automorphism $\sigma \in \text{Aut}(Y)$ is induced by some automorphism $\bar{\sigma} \in \text{Aut}(Z)$.*

Proof. In fact, a proof of this proposition is contained implicitly in the proof of [27, Prop. 2.2]. For the convenience of the reader we reproduce a complete proof here.

Let σ be an automorphism of Y . Then there exist a (non-unique) automorphism $\tilde{\sigma} \in \text{PGL}_6(k)$ such that $\tilde{\sigma}|_Y = \sigma$. Denote $Z' = \tilde{\sigma}(Z)$. Notice that $Z' \cap H = Z \cap H$ where H is a hyperplane which cuts out Y . So it is enough to prove that there exist such automorphism

$\tau \in \text{PGL}_6(k)$ that $\tau(Z) = Z'$ and $\tau|_H = \text{Id}$. Let B be a subgroup of $\text{PGL}_6(k)$ which consists of all elements acting trivially on H . Obviously, G acts transitively on $\mathbb{P}^5 \setminus H$.

The variety Z contains 9 three-dimensional projective subspaces M_{ij} . So Z' also contains nine 3-spaces. Denote them by M'_{ij} . We may assume that $M_{ij} \cap H = M'_{ij} \cap H$ (such intersection are exactly planes of Y). Consider the point

$$v = (1 : 0 : 0 : 0 : 0 : 0), \quad v = \bigcap_{i \neq 1} M_{ij}.$$

We may assume that

$$\bigcap_{i \neq 1} M_{ij} = \bigcap_{i \neq 1} M'_{ij}$$

(otherwise we can apply an automorphism $\tau' \in B$). From $M_{ij} \cap H = M'_{ij} \cap H$ we deduce $M_{ij} = M'_{ij}$ if $i \neq 1$. Let $F' = 0$ be an equation of Z' . Then F' is contained in the intersection of ideals $\bigcap_{i \neq 1} (x_i, y_j)$. It is easy to see that such intersection coincides with the ideal $(x_2x_3, y_1y_2y_3)$, so we may assume that $F' = lx_2x_3 - y_1y_2y_3$ for some non-zero linear polynomial l . Assume that H has an equation $m = 0$. We know that polynomials F' and $x_1x_2x_3 - y_1y_2y_3$ coincide on H and Y is irreducible. Thus one can easily see that $m = l - x_1$ and

$$F' = (x_1 + m)x_2x_3 - y_1y_2y_3$$

(after normalization). Then the automorphism τ acting via

$$x_1 \mapsto x_1 + m, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad y_i \mapsto y_i$$

is contained in B and maps Z to Z' . Thus the composition $\tau \circ \tilde{\sigma}$ is a required automorphism of Z inducing σ . \square

Hence we obtain that the group $\text{Aut}(X)$ is isomorphic to $\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$ if $A^3 = 1$ and \mathfrak{S}_3^2 if $A^3 \neq \pm 1, 0$. But in the last case we have an $\text{Aut}(X)$ -invariant plane $x_1 = x_2 = x_3 = 0$, so the group $\text{Aut}(X)$ is of fiber type by Lemma 2.5. Hence we may assume that X is given by the equations

$$\left\{ x_0x_1x_2 - x_3x_4x_5 = \sum_{i=0}^5 x_i = 0 \right\} \subset \mathbb{P}^5.$$

Now we are going to describe all the possible minimal subgroups of the group $\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$. We will consider $\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$ as a subgroup of \mathfrak{S}_6 with the natural action on coordinates.

Proposition 4.2. *The minimal subgroup G contains with one of the following groups:*

$$\mathfrak{S}_3^2, \quad \mathfrak{C}_3^2 \rtimes \mathfrak{C}_4, \quad \mathfrak{S}_3^2 \rtimes \mathfrak{C}_2,$$

where first group acts transitively on the set of coordinates. Conversely, all such subgroups of $\text{Aut}(X)$ are minimal.

Proof. Recall that there is a natural bijection $l_{ij} \leftrightarrow M_{ij}$ between singular lines and 3-spaces on Z . This bijection induces an $\text{Aut}(X)$ -equivariant bijection between singular points and planes on X . Since the order of the group G is not divisible by 5, a G -orbit of singular point cannot be of length 5. Hence the group G acts transitively on $\text{Sing}(X)$ by Corollary 2.6 and all planes form a unique G -orbit. Since the group $\text{Cl}(X)$ is a free abelian group generated by classes of planes (see [20]), all such subgroups are minimal. A subgroup of $\text{Aut}(X)$ acts transitively on $\text{Sing}(X)$ if and only if it contains the unique subgroup $\mathfrak{C}_3^2 \subset \text{Aut}(X)$.

On the other hand, G cannot act on x_i 's and y_i 's separately since there is no G -invariant planes. Also, the group $\mathfrak{S}_3 \times \mathfrak{C}_3$ has only 1- and 2-dimensional irreducible representations, so all subgroups of G isomorphic to $\mathfrak{S}_3 \times \mathfrak{C}_3$ have a 3-dimensional subrepresentation in our 5-dimensional representation. Thus all such groups have an invariant plane, so they are of fiber type. Thus G is one of the groups from the statement. \square

5. CUBIC THREEFOLDS OF TYPE J11

In this section X is a cubic hypersurface in \mathbb{P}^4 of type J11 satisfying Assumption 2.1 and G is the corresponding minimal subgroup of $\text{Aut}(X)$. We will show that this case does not occur

Proposition 5.1. *The G -variety X with such properties does not exist.*

Proof. Assume that such a G -variety X exists. Any variety of type J11 has 6 nodes and no other singularities. Also X contains three planes Π_1 , Π_2 and Π_3 , each plane contains exactly 4 singularities and every singularity is contained in two planes. Let us denote singularities of X by p_1, \dots, p_6 . We may assume that planes Π_1 , Π_2 and Π_3 contain (p_1, p_2, p_3, p_4) , (p_1, p_2, p_5, p_6) and (p_5, p_6, p_3, p_4) , respectively. These three planes form a hyperplane section of X and have one common non-singular point. We denote this point by p_0 . Let us choose a coordinate system x_0, \dots, x_4 in \mathbb{P}^4 so that

$$\begin{aligned} p_0 &= (1 : 0 : 0 : 0 : 0), \quad p_1 = (0 : 1 : 0 : 0 : 0), \quad p_2 = (1 : 1 : 0 : 0 : 0), \\ p_3 &= (0 : 0 : 1 : 0 : 0), \quad p_4 = (1 : 0 : 1 : 0 : 0), \quad p_5 = (0 : 0 : 0 : 1 : 0), \\ p_6 &= (1 : 0 : 0 : 1 : 0). \end{aligned}$$

The equation of X has the form

$$x_1 x_2 x_3 + x_4 (a x_0 (x_0 - x_1 - x_2 - x_3) + b_1 x_2 x_3 + b_2 x_1 x_3 + b_3 x_1 x_2) +$$

$$+x_4^2(c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3) + dx_4^3 = 0$$

where $a \neq 0$ since p_0 is non-singular point. So we may assume that $a = 1$ without loss of generality (by rescaling x_4). Now we perform the changing of coordinates

$$x_1 \mapsto x_1 - b_1x_4, \quad x_2 \mapsto x_2 - b_2x_4, \quad x_3 \mapsto x_3 - b_3x_4, \quad x_0 \mapsto x_0 - \frac{c_0}{2}x_4$$

to reduce the equation of X to the following form:

$$(3) \quad x_1x_2x_3 + x_4x_0(x_0 - x_1 - x_2 - x_3) + x_4^2(c_1x_1 + c_2x_2 + c_3x_3) + dx_4^3 = 0.$$

Every automorphism of \mathbb{P}^4 which preserves all p_i has the form

$$(4) \quad x_i \mapsto x_i + \alpha_i x_4$$

for some α_i in our coordinate system. It is easy to see that such an automorphism preserves X if and only if the automorphism is trivial.

Hence we have the canonical injection

$$\text{Aut}(X) \rightarrow \mathfrak{C}_2^3 \rtimes \mathfrak{S}_3 \subset \mathfrak{S}_6$$

to the group of permutations of singular points. We know by Corollary 2.6 that $\text{Aut}(X)$ acts transitively on the set of singular points. Notice that the involution

$$\sigma : (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (-x_0 + x_1 + x_2 + x_3 : x_1 : x_2 : x_3 : x_4)$$

always acts on X . So the action of $\text{Aut}(X)$ on $\text{Sing}(X)$ is transitive if and only if $\text{Aut}(X)$ contains an element of order 3. All elements of order 3 in $\mathfrak{C}_2^3 \rtimes \mathfrak{S}_3$ are conjugate, so we may assume that $\text{Aut}(X)$ contains the automorphism acting on singular points as follows:

$$p_1 \mapsto p_3, \quad p_3 \mapsto p_5, \quad p_5 \mapsto p_1, \quad p_2 \mapsto p_4, \quad p_4 \mapsto p_6, \quad p_6 \mapsto p_2.$$

Obviously, such automorphism is the composition of the cyclic permutations of coordinates x_1 , x_2 and x_3 and a morphism of type (4). Such automorphism acts on X if and only if it is cyclic permutation of coordinates (it follows from vanishing of some coefficients) and $c_1 = c_2 = c_3$. In this case $\mathfrak{S}_3 \times \mathfrak{C}_2$ acts on X automatically. If $\text{Aut}(X) = \mathfrak{C}_2 \times \mathfrak{S}_3$ then we have an $\text{Aut}(X)$ -invariant plane $x_1 + x_2 + x_3 = x_4 = 0$ in \mathbb{P}^4 , this is impossible due to Assumption 2.1. There are no intermediate groups between $\mathfrak{C}_2 \times \mathfrak{S}_3$ and $\mathfrak{C}_2^3 \rtimes \mathfrak{S}_3 \simeq \mathfrak{S}_4 \times \mathfrak{C}_2$. Hence the last thing we need to check is when the automorphism which changes p_1 and p_2 and fixes all other p_i 's acts on X . This automorphism can be explicitly written as

$$\sigma' : (x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_1 + x_4 - x_0 : x_1 : x_4 - x_2 : x_4 - x_3 : x_4)$$

(this is the only automorphism which acts in the right way on singularities and preserves vanishing of some coefficients) and the image of X under this morphism is the variety with the equation

$$x_1x_2x_3 + x_4x_0(x_0 - x_1 - x_2 - x_3) + x_4^2(c_1x_1 + (1 - c_2)x_2 + (1 - c_3)x_3) + (d + c_2 + c_3 - 1)x_4^3 = 0.$$

This variety coincides with X if and only if $c_1 = c_2 = c_3 = \frac{1}{2}$.

Consider the subgroup of $\mathrm{GL}_5(k)$ generated by σ , σ' (where we replace all colons by commas) and permutations of coordinates x_1 , x_2 and x_3 . This group can be regarded as a lift of $\mathrm{Aut}(X) \subset \mathrm{PGL}_5(k)$ to $\mathrm{GL}_5(k)$ and is isomorphic to $\mathrm{Aut}(X)$. Hence we have a five-dimensional representation of $\mathrm{Aut}(X)$. Since the group $\mathrm{Aut}(X) \simeq \mathfrak{S}_4 \times \mathfrak{C}_2$ has only one-, two- and three-dimensional irreducible representations, there is a three-dimensional subrepresentation which gives us an invariant plane in \mathbb{P}^4 . One can easily see that the plane with the equations

$$x_4 = x_0 - \frac{1}{2}(x_1 + x_2 + x_3) = 0$$

is $\mathrm{Aut}(X)$ -invariant. Thus $\mathrm{Aut}(X)$ is of fiber type by Lemma 2.5. Hence the required G -variety X does not exist. \square

6. CUBIC THREEFOLDS OF TYPE J9

In this section X is a cubic hypersurface in \mathbb{P}^4 of type J9 satisfying Assumption 2.1 and G is the corresponding minimal subgroup of $\mathrm{Aut}(X)$. In this case X contains six singularities in general position (see Proposition 2.9 and Lemma 2.8). Hence the group $\mathrm{Aut}(X)$ is embedded canonically into the group of permutations of singularities \mathfrak{S}_6 . Below we regard $\mathrm{Aut}(X)$ as a subgroup of \mathfrak{S}_6 . We may assume that singularities of X are

$$p_1 = (1 : 0 : 0 : 0 : 0), \quad p_2 = (0 : 0 : 1 : 0 : 0), \quad p_3 = (0 : 0 : 0 : 0 : 1),$$

$$p_4 = (0 : 1 : 0 : 0 : 0), \quad p_5 = (0 : 0 : 0 : 1 : 0), \quad p_6 = (1 : 1 : 1 : 1 : 1).$$

The equation of X has a form

$$(5) \quad \sum_{0 \leq i < j < k \leq 4} a_{ijk} x_i x_j x_k = 0,$$

where for every $0 \leq t \leq 4$ the sum of all coefficients which has t as one of the indices equals to zero. Since the group $\mathrm{Aut}(X)$ is neither linearizable nor of fiber type, it must act transitively on $\mathrm{Sing}(X)$ by Corollary 2.6.

Proposition 6.1. *In some coordinate system X is given by the equation (5) with*

$$(6) \quad a_{024} = A, \quad a_{012} = -a_{123} = a_{234} = B, \quad a_{014} = -a_{013} = a_{023} = C, \\ a_{034} = a_{124} = -a_{134} = D, \quad A + B + C + D = 0,$$

where A, B, C and D are nonzero numbers. Moreover:

- (i) if $A = B = -C = -D$ or $A = -B = C = -D$ or $A = -B = -C = D$ then $\text{Aut}(X) \simeq \mathfrak{S}_5$;
- (ii) if $B = C = D$ then $\text{Aut}(X) \simeq \mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$, $\text{Aut}(X)$ is a normalizer of a Sylow 3-subgroup in \mathfrak{S}_6 ;
- (iii) if $A = -B$ or $A = -C$ or $A = -D$ then $\text{Aut}(X) \simeq \mathfrak{S}_4$;
- (iv) if $C = D$ or $B = C$ or $B = D$ then $\text{Aut}(X) \simeq \mathfrak{D}_{12}$;
- (v) in all other cases $\text{Aut}(X) \simeq \mathfrak{S}_3$.

Proof. At first we notice that all coefficients a_{ijk} are nonzero. Indeed, suppose that $a_{012} = 0$. Consider the projection from the point p_3 to the hyperplane $x_4 = 0$. The image of the exceptional divisor is a curve, which is the intersection of the quadric

$$\sum_{0 \leq i < j \leq 3} a_{ij4} x_i x_j = 0$$

and the cubic

$$a_{013} x_0 x_1 x_3 + a_{023} x_0 x_2 x_3 + a_{123} x_1 x_2 x_3 = 0.$$

But such cubic is reducible, so X cannot be of type J9 (see [12] for details).

Every subgroup of \mathfrak{S}_6 which acts transitively on the set of singularities contains an element of order 3 acting with two orbits of length 3 (one can easily see it from the table of subgroups of \mathfrak{S}_6 [29]). Without loss of generality we may assume that this element is $(1, 2, 3)(4, 5, 6)$. Moreover, we can write down its action explicitly:

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_4 - x_3 : -x_3 : x_0 - x_3 : x_1 - x_3 : x_2 - x_3).$$

This automorphism maps X to the variety with the equation

$$a_{234} x_0 x_1 x_2 + a_{134} x_0 x_1 x_3 + a_{124} x_0 x_1 x_4 + a_{034} x_0 x_2 x_3 + a_{024} x_0 x_2 x_4 + \\ + a_{014} x_0 x_3 x_4 - a_{012} x_1 x_2 x_3 - a_{013} x_1 x_2 x_4 - a_{023} x_1 x_3 x_4 - a_{123} x_2 x_3 x_4 = 0.$$

This cubic coincides with X exactly when condition (6) holds. Moreover, the automorphism acting on singular points as $(1, 4)(2, 6)(3, 5)$ also acts on X in this case. Hence if the condition (6) holds then $\text{Aut}(X)$ acts transitively on the set of singular points.

Assume that $\text{Aut}(X)$ contains

$$H \simeq \mathfrak{S}_3 = \langle (1, 2, 3)(4, 5, 6), (1, 4)(2, 6)(3, 5) \rangle$$

as a proper subgroup. There are exactly seven groups H' such that $H \subsetneq H'$ and there are no intermediate groups between H and H' :

- three groups isomorphic to \mathfrak{D}_{12} . They are generated by H and either $h_{11} = (1, 4, 2, 5, 3, 6)$ or $h_{12} = (1, 5, 2, 6, 3, 4)$ or $h_{13} = (1, 6, 3, 2, 5, 4)$ respectively;
- a unique group isomorphic to $\mathfrak{S}_3 \times \mathfrak{C}_3$ which is generated by H and $h_2 = (1, 2, 3)$;
- three subgroups isomorphic to \mathfrak{S}_4 . They are generated by H and either $h_{31} = (1, 2, 4, 5)$ or $h_{32} = (1, 2, 5, 6)$ or $h_{33} = (1, 2, 6, 4)$ respectively.

Direct computations show the following:

- h_{11} preserves $X \iff C = D$;
- h_{12} preserves $X \iff B = C$;
- h_{13} preserves $X \iff B = D$;
- h_2 preserves $X \iff B = C = D$;
- h_{31} preserves $X \iff A = -D$;
- h_{32} preserves $X \iff A = -B$;
- h_{33} preserves $X \iff A = -C$.

If $B = C = D$ then, obviously, the group $\text{Aut}(X)$ contains also elements $(1, 2)$ and $(4, 5)$ and does not contain h_{31} , so it is equal to $\mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$.

Now we assume that $\text{Aut}(X) \supsetneq \mathfrak{D}_{12} = \langle H, h_{11} \rangle$ and $\text{Aut}(X)$ does not contain h_2 . Then one of the two possibilities occurs (see [8]): either

$$\text{Aut}(X) \simeq \mathfrak{S}_4 \times \mathfrak{C}_2 = \langle \mathfrak{D}_{12}, h_{32} \rangle$$

or

$$\text{Aut}(X) \simeq \mathfrak{S}_5 = \langle \mathfrak{D}_{12}, h_{31} \rangle.$$

In the first case we have

$$C = D, \quad A = -B, \quad A + B + C + D = 0,$$

thus $C = D = 0$, a contradiction. In the second case

$$A = B = -C = -D.$$

All other groups isomorphic to \mathfrak{D}_{12} can be considered in the same way.

Now we assume that $\text{Aut}(X) \supsetneq \mathfrak{S}_4 = \langle H, h_{31} \rangle$. Then one of the three possibilities occurs: either

$$\text{Aut}(X) \simeq \mathfrak{S}_4 \times \mathfrak{C}_2 = \langle \mathfrak{S}_4, h_{12} \rangle,$$

or

$$\text{Aut}(X) \simeq \mathfrak{S}_5 = \langle \mathfrak{S}_4, h_{11} \rangle,$$

or

$$\text{Aut}(X) \simeq \mathfrak{S}_5 = \langle \mathfrak{S}_4, h_{13} \rangle.$$

All these cases have already been considered. \square

Remark 6.2. One can easily verify that a variety given by equation (i) or (ii) of Proposition 6.1 is of type J9.

Proposition 6.3. *In the notation of the previous proposition $\text{Aut}(X)$ cannot be isomorphic to \mathfrak{S}_3 , \mathfrak{S}_4 or \mathfrak{D}_{12} . If $\text{Aut}(X) \simeq \mathfrak{S}_5$, then $G = \text{Aut}(X)$. If $\text{Aut}(X) \simeq \mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$, then $G = \text{Aut}(X)$ or a unique subgroup isomorphic to \mathfrak{S}_3^2 acting transitively on $\text{Sing}(X)$.*

Proof. Here we use the notation of the proof of the previous proposition. Assume that $\text{Aut}(X)$ equals to H or $\mathfrak{D}_{12} = \langle H, h_{11} \rangle$ (we do not lose generality since we may consider groups up to conjugation in \mathfrak{S}_6). Consider the point $(1 : -1 : 0 : -1 : 1)$. The $\text{Aut}(X)$ -orbit of this point consists of three points

$$(1 : -1 : 0 : -1 : 1), (2 : 1 : 2 : 0 : 1), (1 : 0 : 2 : 1 : 2).$$

They are not collinear, so the plane through them is the $\text{Aut}(X)$ -invariant plane. In the case $\text{Aut}(X) \simeq \mathfrak{S}_4 = \langle H, h_{31} \rangle$ we may consider the point $(1 : 1 : 1 : 1 : 2)$ with the same property. In any case $\text{Aut}(X)$ is of fiber type by Lemma 2.5, it is a contradiction with Assumption 2.1.

Let π be a projection to the hyperplane $L = \{x_0 = 0\}$ from the point p_1 . The exceptional divisor of π is mapped to two rational curves Γ_1 and Γ_2 and singular points $p_i, i > 1$, are mapped to five intersection points $\Gamma_1 \cap \Gamma_2$ (see [12] for details). Denote $r_i = \pi(p_i)$, $i > 1$. Let S_i be a preimage of Γ_i . Then the class of S_1 and $\frac{1}{2}K_X$ generate $\text{Cl}(X)$ (see [12]) and $S_1 + S_2 \sim -K_X$.

Assume that $\text{Aut}(X) \simeq \mathfrak{S}_5$. Since G acts transitively on $\text{Sing}(X)$ and is not contained in \mathfrak{S}_4 nor \mathfrak{D}_{12} , the group G must be equal either to \mathfrak{A}_5 or to \mathfrak{S}_5 . The group \mathfrak{A}_5 cannot act on the lattice $\text{Cl}(X)$ with $\text{rk Cl}(X)^G = 1$, so $G = \text{Aut}(X)$. Now we will prove that $\text{Aut}(X)$ acts minimally on X . Let $F \simeq \mathfrak{C}_5 \rtimes \mathfrak{C}_4$ be the stabilizer of the point p_1 . Then π is an F -equivariant map. A subgroup $\mathfrak{C}_5 \subset F$ acts non-trivially on $\text{Sing}(X)$. Thus the group F cannot act on rational curve and need to permute Γ_i 's. Hence F interchanges S_1 and S_2 , so F acts minimally on X .

Finally, consider the case $\text{Aut}(X) \simeq \mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$. The stabilizer of the point p_1 is isomorphic to $\mathfrak{S}_3 \times \mathfrak{C}_2 \simeq \mathfrak{D}_{12}$ and acts faithfully on $\text{Sing}(X) \setminus \{p_1\}$ with two orbits of length 2 and 3 respectively. Thus the stabilizer of p_1 cannot act on Γ_1 and Γ_2 separately and the group $\text{Aut}(X)$ is minimal for the same reason as in the previous case.

The kernel of the homomorphism

$$\text{Aut}(X) \rightarrow \text{Aut}(\text{Cl}(X)) \simeq \text{GL}_2(\mathbb{Z})$$

is a subgroup of $\text{Aut}(X)$ of index 2. There are exactly three subgroups of index 2 in $\text{Aut}(X)$: the group $G_1 \simeq \mathfrak{S}_3^2$ which does not act transitively on $\text{Sing}(X)$, the group $G_2 \simeq \mathfrak{S}_3^2$ acting transitively on $\text{Sing}(X)$ and the group $G_3 \simeq \mathfrak{C}_3^2 \rtimes \mathfrak{C}_4$. The first subgroup has the same stabilizer of p_1 as $\text{Aut}(X)$, so this subgroup cannot be the kernel.

Assume that $\delta = (1, 4)(2, 6)(3, 5) \in G_2$ acts trivially on $\text{Cl}(X)$. One can easily check that the set of δ -invariant points of X consists of an elliptic curve and three points. Let $\tau : \tilde{X} \rightarrow X$ be a δ -equivariant small resolution of singularities. Then the contraction of some extremal ray gives us a δ -equivariant \mathbb{P}^1 -fibration over \mathbb{P}^2 (see [20, Theorem 5.2]). If δ acts trivially on the base then there is a divisor of fixed points on \tilde{X} and the same is true for X since τ is small. A contradiction. If δ acts non-trivially on the base then the set of δ -invariant points on \mathbb{P}^2 consists a line and a point. Thus there are two lines of fixed points on \tilde{X} . These lines cannot be contracted by τ since there are no δ -invariant singular points on X . This is impossible. Thus δ acts non-trivially on $\text{Cl}(X)$. Hence G_3 is the required kernel.

The group G is a subgroup of $\text{Aut}(X)$ acting transitively on $\text{Sing}(X)$ which is not a subgroup of $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_4$. Also G is not contained in any subgroup of $\text{Aut}(X)$ isomorphic to \mathfrak{D}_{12} for the same reason as $\text{Aut}(X) \not\simeq \mathfrak{D}_{12}$. Thus either G is conjugate to $\mathfrak{S}_3 \times \mathfrak{C}_3 = \langle H, (1, 2, 3) \rangle$ or G coincides with G_2 or $\text{Aut}(X)$. But in the first case the G -orbit of the point $(\omega : \omega^2 - 1 : -2 : \omega - 1 : \omega^2)$ where ω is a cubic root of unity has length 3. \square

7. FIVE SINGULAR POINTS

In this section we assume that X is a cubic hypersurface in \mathbb{P}^4 with exactly five singular points in general position satisfying Assumption 2.1 and G is the corresponding minimal subgroup of $\text{Aut}(X)$.

Proposition 7.1. *The variety X can be given in some coordinate system by the equation*

$$\begin{aligned} & (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_0 + x_4x_0x_1) + \\ & + a(x_0x_1x_3 + x_1x_2x_4 + x_2x_3x_0 + x_3x_4x_1 + x_4x_0x_2) = 0, \quad a^3 = 1. \end{aligned}$$

The group $\text{Aut}(X)$ is isomorphic to \mathfrak{S}_5 if $a = 1$ and to \mathfrak{A}_5 otherwise. The group G is isomorphic either to \mathfrak{S}_5 or \mathfrak{A}_5 or $\mathfrak{C}_5 \rtimes \mathfrak{C}_4$. All the singularities of X are nodes.

Proof. We assume that singular points are $p_i = \{x_j = \delta_i^j\}$. The equation of X in such a coordinate system is

$$\sum_{0 \leq i < j < k \leq 4} a_{ijk} x_i x_j x_k.$$

The action of $\text{Aut}(X)$ on the set of singularities of X induces a homomorphism $\text{Aut}(X) \rightarrow \mathfrak{S}_5$ with transitive image, so the image contains an element of order 5. Without loss of generality we may assume that the image contains the element $(1, 2, 3, 4, 5)$. Obviously, every element of $\text{PGL}_5(k)$ which permutes cyclically singularities of X is the composition of some diagonal map and the cyclic permutation of coordinates. All such elements are conjugate to each other by diagonal map. This can be seen from the explicit equations for elements of corresponding diagonal matrix. Moreover such diagonal change of coordinates is unique. As a consequence, we see that the homomorphism $\text{Aut}(X) \rightarrow \mathfrak{S}_5$ is an embedding. Thus we may assume that $\text{Aut}(X)$ contains a cyclic permutation of coordinates and

$$a_{012} = a_{123} = a_{234} = a_{034} = a_{014}, \quad a_{013} = a_{124} = a_{023} = a_{134} = a_{024}.$$

Note that all coefficients are nonzero, otherwise the singularities of X are not isolated. Thus we may assume that $a_{012} = 1$ and $a_{013} = a$ for some nonzero a . Note that if $a = -1$ then X has one more singularity at the point $(1 : 1 : 1 : 1 : 1)$, so $a \neq -1$. Obviously, \mathfrak{D}_{10} acts on X naturally. Up to conjugation there are three subgroups in \mathfrak{S}_5 containing \mathfrak{D}_{10} as a proper subgroup: $\mathfrak{C}_5 \rtimes \mathfrak{C}_4$, \mathfrak{A}_5 and \mathfrak{S}_5 . The group

$$\mathfrak{C}_5 \rtimes \mathfrak{C}_4 = \langle \mathfrak{D}_{10}, (2, 3, 5, 4) \rangle$$

acts on X if and only if $a = 1$ and, obviously, in this case $\text{Aut}(X) = \mathfrak{S}_5$. The group $\mathfrak{A}_5 = \langle \mathfrak{D}_{10}, (1, 2, 3) \rangle$ acts on X if and only if $a^3 = 1$.

Now we assume that $a^3 \neq 1$. Consider points $(1 : \zeta : \zeta^2 : \zeta^3 : \zeta^4)$ and $(1 : \zeta^4 : \zeta^3 : \zeta^2 : \zeta)$, where ζ is a primitive fifth degree root of unity. They form a \mathfrak{D}_{10} -orbit and lie on X . The line l passing through them also lies on X . Indeed, otherwise l has the same local intersection numbers with X at both of this points, so there is the third intersection point of l and X . Such a point must be invariant under the action of \mathfrak{D}_{10} , which can be true only if this point is $(1 : 1 : 1 : 1 : 1)$. But such a point does not lie on X since $a \neq -1$, a contradiction. Hence the projection from l gives us conic fibration, which is equivalent to the some Mori fibration. Thus $a^3 = 1$.

The group G acts transitively on $\text{Sing}(X)$ and is not contained in \mathfrak{D}_{10} , so it must be one of the groups from the statement. In both cases ($a = 1$ or a is a primitive cubic root of unity) X has only ordinary

double points as singularities, so $\text{Cl}(X)$ is generated by the class of hyperplane section (see [12]). Hence all the subgroups of $\text{Aut}(X)$ are minimal automatically. One can easily check the last statement of the proposition by a direct computation. \square

8. FOUR SINGULAR POINTS

Proposition 8.1. *Under the assumptions of 2.1 the variety X cannot have less than 5 singular points.*

Proof. Assume that X has less than 5 singular points. Then X has exactly 4 singularities in general position and $\text{Aut}(X)$ acts on $\text{Sing}(X)$ transitively by the Corollary 2.6. Let S be a hyperplane section of X passing through all the singular points. It is a singular cubic surface (maybe reducible) with at least four singular points in general position. Due to [4] such a surface must be either the special cubic surface with four nodes or reducible. In the second case S must be either a union of three planes with unique common point or a union of a quadric cone whose vertex is a singular point of X and a plane, otherwise singularities of X cannot be in general position. In both cases $\text{Aut}(X)$ cannot act transitively on $\text{Sing}(X)$: if S is a union of three planes either the common point of planes is a distinguished singularity of X or there is a distinguished plane that contains exactly two singularities of X ; if S is a union of a quadric cone and a plane then the vertex of the cone is a distinguished singularity of X . The cubic surface with four singularities of type A_1 in suitable coordinate system has the equation

$$F(x_0, x_1, x_2, x_3) = x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0.$$

So in some coordinate system X has the equation [4]

$$F(x_0, x_1, x_2, x_3) + x_4 \sum_{0 \leq i < j \leq 3} a_{ij}x_i x_j + x_4^2 \sum_{0 \leq i \leq 3} b_i x_i + c x_4^3 = 0.$$

After some changing of coordinates as follows

$$(7) \quad x_i \mapsto (x_i + \alpha_i x_4), \quad \alpha_4 = 0$$

with appropriate α_i we obtain

$$(8) \quad a_{02} = a_{13} = A, \quad a_{01} = a_{23} = B, \quad a_{12} = a_{03} = C, \quad A + B + C = 0$$

for some numbers A, B, C . Moreover, such changing of coordinates is unique (one can easily see this by solving the corresponding system of equations for α_i). Every element $\sigma \in \text{PGL}_5(k)$ which preserves $\text{Sing}(X)$ can be represented as a composition $\sigma = \sigma_3 \circ \sigma_2 \circ \sigma_1$ where σ_1 permute the coordinates x_0, x_1, x_2, x_3 , σ_2 acts by a multiplying x_4 with some number and σ_3 looks like (7). Automorphisms σ_1 and σ_2 preserve

the property (8), while σ_3 preserves it if and only if σ_3 is trivial. So, every element of $\text{Aut}(X)$ is a composition $\sigma_2 \circ \sigma_1$ and consequently it preserves the subspace $x_4 = x_0 + x_1 + x_2 + x_3 = 0$. Thus the group $\text{Aut}(X)$ is of fiber type by Lemma 2.5. \square

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